3 Statistical Description of Macroscopic Systems

Any macroscopic system consists of an enormously large number of atomic particles. It would be an impossibly complex task to describe in detail every individual particle in such a system, nor would this be relevant if one is primarily interested in the large-scale properties of a system. Hence one can usefully resort to a *statistical* description of macroscopic systems, i.e., to a much less detailed description in terms of probabilities. The present chapter discusses how this can be done and thereby provides a rather simple, yet very powerful, method for understanding the properties of macroscopic systems.

A. Statistical descriptions

There are some systems which are sufficiently simple and well-understood that one can make definite predictions about their behavior. For example, consider the system consisting of the sun and the planets. If the positions and velocities of these bodies are known at any instant, a knowledge of mechanics allows one to predict the positions and velocities of these bodies at any other time. Astronomical predictions of this kind are, indeed, remarkably precise.

On the other hand, there are many other systems where such definite predictions can *not* be made because the systems are too complex or because not enough information is available about them. For example, consider a seemingly simple system consisting of a pair of coins. If one knew precisely just how these coins are tossed by a person's hand, one might be able to use Newtonian mechanics to predict how these coins land on the floor (e.g., how many "heads" and "tails" are produced by the toss). But one ordinarily does *not* know precisely how these coins are tossed; indeed, it would be difficult or impossible to obtain such precise information. Thus it is *not* possible to predict the outcome of a toss with any certainty.

However, some useful predictions can still be made. For, although one may not be able to make definite predictions with certainty, one can still predict the *probabilities* of various outcomes.

The following paragraphs outline how one can achieve such *statistical* predictions, i.e., predictions that specify probabilities. In particular, we shall specify the essential ingredients necessary for a statistical approach and shall illustrate this approach with a simple example (the tossing of a pair of coins). However, the approach is generally applicable to much more complex cases (e.g., to gambling casinos or insurance companies) and will later be applied to macroscopic systems consisting of very many particles.

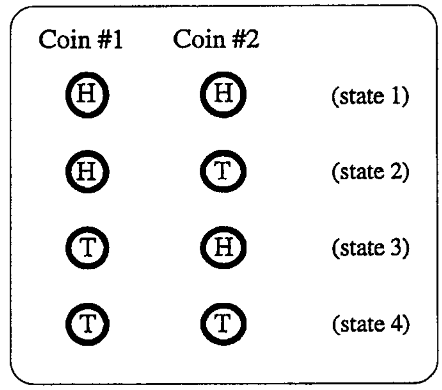


Fig. A-1. Possible states of a system consisting of two coins. (Heads is indicated by H, tails by T.)

Specification of states

To discuss a system, one must first be able to specify the possible states of the system — so that one can specify the possible outcomes of any experiment involving the system.

For example, consider the system consisting of two coins. Then any coin lying on a flat surface can be in either one of two possible states, i.e., its upper side can either show heads or tails. The state of the system consisting of *both* coins can then be specified by specifying the state of each coin in the system. The system can thus be in any one of the four possible states indicated in Fig. A-1.

Statistical assembly and probabilities

As already mentioned, it is not possible to predict the outcome resulting from any particular toss of the two coins. However, suppose that, instead of focusing attention on a single system of two such coins, one considers a *statistical assembly* consisting of a very large number $\mathcal{N}$ of such systems, all observed under the same known conditions and tossed in similar ways. (See Fig. A-2.) Then some number $\mathcal{N}_r$ of these systems will end up in any particular state r (where r can denote any of the possible states 1, 2, 3, ...). One can *not* predict which particular systems end up this state. However, one may readily be able to predict what *fraction* $P_r = \mathcal{N}_r/\mathcal{N}$ of the systems in the assembly end up in any particular state r. This fraction is called the probability P_r of finding the system in the particular state r.

We use here the word "assembly" instead of the more fancy technical word "ensemble".

Def: **Probability:** Suppose that $\mathcal{N}_r$ systems in a statistical assembly of $\mathcal{N}$ similar systems are in a particular state r. The probability P_r that a system in this assembly is in the state r is then defined by the ratio

$$P_r = \mathcal{N}_r/\mathcal{N} \quad \text{(if } \mathcal{N} \text{ is very large).} \tag{A-1}$$

(This probability does not depend on the exact number $\mathcal{N}$ of systems in the assembly if this number is sufficiently large.)

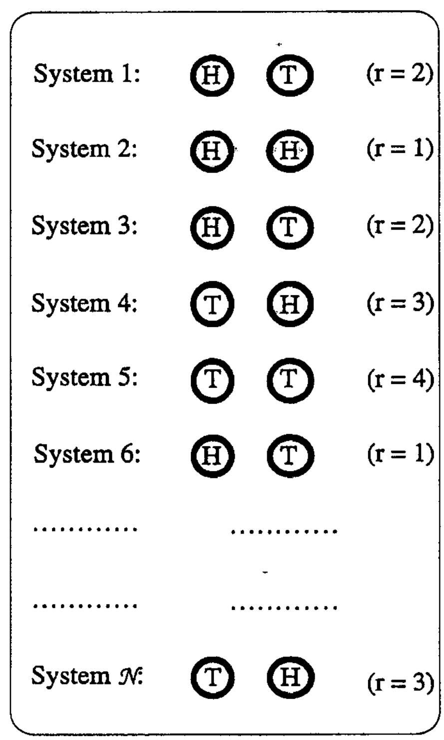


Fig. A-2. A statistical assembly of $\mathcal{N}$ systems, each consisting of two coins. (The state of each system is indicated in parentheses.)

Example

To obtain statistical data about a system consisting of two particular coins, one might consider an assembly of 10,000 such systems and toss the coins in each such system.. Suppose that the numbers of systems found in each of the four possible states are those indicated in the second column of Fig. A-3. The figures in the last column then indicate the probabilities that the system is found in each of these states.

Sum of probabilities for all states. The sum of the numbers of systems found in all the possible states r = 1, 2, 3, ...is, of course, must be equal to the total number of systems in the assembly. Thus

$$\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \ldots = \mathcal{N}.$$

If one divides both sides by $\mathcal{N}$, this is equivalent to the statement that

$$P_1 + P_2 + P_3 + \ldots = 1. \quad \text{(A-2)}$$

In other words, the sum of the probabilities of finding a system in all of its possible states must always be equal to 1.

Probabilities depend on the statistical assembly of interest. Probabilities depend crucially on the particular statistical assembly being considered. For example, the probability that a person will be killed in a car crash depends on whether the person is considered as a member of the statistical assembly of all drivers, or as a member of the assembly of all drivers below the age of 21 years. (This is why auto insurance rates are different for populations of different ages.)

Assemblies for time-independent situations.
A specially simple situation is one which is time-independent so that the probabilities don't change with time. (For example, the same probabilities would be observed no matter when the coins are tossed.) Instead of observing a statistical assembly of $\mathcal{N}$ systems at any one time, one may then just as well consider a statistical assembly consisting of a single system observed $\mathcal{N}$ times in succession.

State		$\mathcal{N}_r$	P_r
r = 1	(H, H)	2,600	0.26
r = 2	(H, T)	2,500	0.25
r = 3	(T, H)	2,500	0.25
r = 4	(T, T)	2,400	0.24
Sum		$\mathcal{N}$= 10,000	1.00

Fig. A-3. Results obtained by tossing a set of two coins. (The observed statistical assembly consisted of 10,000 such sets of coins.)

Statistical postulates

Any successful theory should allow one to make many predictions on the basis of a few fundamental postulates. A *statistical* theory too must be based on some fundamental postulates. The only difference is that such a statistical theory does not predict particular events with certainty, but aims to predict the *probabilities* of various events.

For example, suppose that one is tossing coins whose two sides are mechanically indistinguishable (e.g., one side is not any heavier than the other). Then one might reasonably make the assumption (accepted as fundamental statistical *postulate)* that each such coin is equally probable to land heads or tails.

Statistical calculations

Once one has adopted such a fundamental statistical postulate, one can use it to calculate various probabilities of interest. These predicted probabilities can, of course, be compared with the probabilities observed in appropriate assemblies of systems. If many such predictions are well verified, one may have increasing faith in the validity of the statistical postulate. Otherwise, the postulate may need to be modified.

For example, according to the previously mentioned theoretical postulate, each coin should be equally likely to land heads or tails. Applied to a system of two tossed coins, this postulate implies that each of the four possible states should be equally probable. The data listed in Fig. A-3 do not quite agree with this prediction. Hence one can conclude that the postulate is not quite valid in the case if the particular pair of coins for which these data were obtained.

Problems

[A-1] ***Tossing three coins***

Three coins are tossed in a game of chance. To determine the probabilities of various outcomes resulting from tosses of the three coins, one can imagine tossing a large number of such sets of coins and can determine in what fraction of cases various outcomes occur.

(a) Any possible outcome of a toss (i.e., any possible state of the system of three coins) can be specified by specifying whether each coin shows heads or tails. What is the total number of possible states that can result from a toss? *<h-3>*

(b) List all of the possible states of the system of three coins. Specify each state by indicating, for each coin, whether it shows heads or tails. (For example, the state where each coin shows heads can be indicated by writing HHH.)

(c) How many of these states are such that all three coins show heads? How many of these states are such that two coins show heads (and the remaining one coin shows tails)? How many of these states are such that only one coin shows heads? How may are such that no coin shows heads?

(d) Suppose that one knows that each individual coin is equally likely to land heads or tails. When tossing the three coins, what then is the probability of obtaining three heads? What is the probability of obtaining two heads? What is the probability of obtaining one heads? What it the probability of obtaining no heads?

(e) W is the probability of obtaining three tails? What is the probability of obtaining two tails? What is the probability of obtaining one tails? What it the probability of obtaining no tails? *<a-4>*

[A-2] ***Rolling two dice***

A common game of chance involves the rolling of two dice, each of which can show one of its six faces (which are labeled by 1, 2, 3, 4, 5, or 6 dots, respectively).

(a) What is the total number of possible outcomes resulting from the rolling of a pair of such dice? *<h-5>*

(b) The score resulting from rolling the dice is obtained by adding the number of dots on the shown faces of the dice. How many possible outcomes are there in which the score is 1? In which the score is 2? In which the score is 3? In which the score is 4? *<a-7>*

(c) If the dice are fair (i.e., if no one has tried to cheat by "loading" the dice), each rolled die is equally likely to show each of its six faces. Under these conditions, what is the probability P_n that a score of n is obtained when rolling the dice? Calculate this probability for all values of n between n = 1 and n = 12.

(d) What is the most probable value of this score? What is the probability that this value will be obtained as a result of rolling the dice? *<a-2>*

[A-3] ***Mean values***

Consider a statistical assembly consisting of a large number $\mathcal{N}$ of similar systems. Suppose that $\mathcal{N}_1$ of these systems are in the state 1 where some quantity q has the value q_1, that $\mathcal{N}_2$ of these systems are in the state 2 where the quantity q has the value q_2, etc. Then the mean value (or average value) $\bar{q}$ of the quantity q is defined to be

$$\bar{q} = (\mathcal{N}_1 q_1 + \mathcal{N}_2 q_2 + \mathcal{N}_3 q_3 + \ldots)/\,\mathcal{N}$$

where the sum extends over all possible states of the system.

Express the mean value $\overline{q}$ solely in terms of the probabilities P_1, P_2, ... that the system is in its various possible states and the values q_1, q_2, ... of the quantities in these states. <*a-3*>

[A-4] *Probabilities of choosing hiking trails*

When children are taken to a day camp, each child is given a choice of hiking along one of three trails. The first of these trails is 3.4 miles long, the second is 2.2 miles long, and the third is 2.8 miles long. Experience in this day camp indicates that the probability that a child chooses the first of these trails is 0.32 and the probability that a child chooses the second of these trails is 0.43.

(a) What is the probability that a child choose the third of these trails?
(b) What is the probability that a child chooses either the first or the second of these trails?
(c) What is the mean distance hiked by a child? <*h-1*> <*h-9*> <*a-10*>

B. Statistical theory of macroscopic systems

The statistical approach outlined in the preceding section, and illustrated in the case of the pair of coins, can equally well be applied to any macroscopic system consisting of many atomic particles. The following paragraphs discuss how this can be done by proceeding in the same way as in the preceding section.

Specification of states

The state of a macroscopic system can be described in detail by specifying its *basic state*:

Def: ***Basic state of a system:*** The state of the system, described from an atomic point of view. (B-1)

Such a basic state can be specified by specifying the position and velocity of every atom in the system. For example, this can be done by specifying the three position coordinates x, y, z of every particle and the three velocity components v_x, v_y, v_z of every particle. (Needless to say, this requires the specification of an enormously large number of quantities. But, in principle, this can be done.)

From a more accurate quantum-mechanical point of view, a basic state of a system is a quantum state specified by a set of quantum numbers.

Countability of basic states. The possible states of the two-coin system could readily be counted so that it was easy to talk about the probability of each such state. How can one similarly count the possible basic states of a macroscopic system although positions and velocities are continuous quantities?

For example, consider a particle's position coordinate x. To specify this position coordinate with some limited precision, we may ignore distinctions between neighboring values of x lying within some small elementary range of size δx. All these values may then be lumped together and simply designated by a common value characterizing this range. [For instance, suppose that we do not care to measure x more precisely than within a range δx of 1 micron (i.e., 10^{-6} meter). Then we can simply say that x = 0 whenever it actually lies anywhere in the range between −0.5 micron and + 0.5 micron. Similarly, we can simply say that x = 3 micron whenever it actually lies anywhere in the range between 2.5 micron and 3.5 micron.] As illustrated in Fig. B-1, all the values of x can then be adequately specified by the values labeling these elementary ranges (e.g., the values 0 micron, 1 micron, 2 micron, etc.). Since these values are discrete and countable, the values of x have thus been *digitized.*

Information is digital if it can be represented by integers. (All information in modern computers is expressed in this form.)

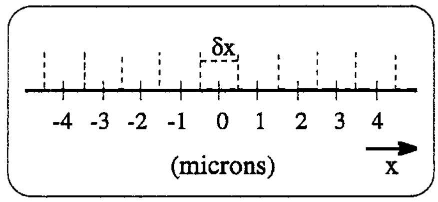


Fig. B-1. Values of x grouped together into small elementary ranges of size δx. Each such range can be specified by its central value.

We can deal similarly with a particle's velocity. For example, suppose that we do not care to measure any particle's velocity component v_x more precisely than within a small range δv_x of 1 cm/s. Then we can simply say that $v_x = 0$ whenever it actually lies anywhere in the range between –0.5 cm/s and + 0.5 cm/s. Similarly, we can simply say v_x = 3 cm/s whenever v_x actually lies anywhere in the range between 2.5 cm/s and 3.5 cm/s. Each such small range can then be labeled by a single value representing indiscriminately any of the values of v_x in this range. These labeling values (e.g., 1 cm/s, 2 cm/s, 3c cm/s, etc.).can then be used to specify the values of v_x within the desired precision. Since these labeling values are discrete and countable, the values of v_x have thus been digitized.

The preceding comments indicate how the basic states of a macroscopic system can be specified (and how this can be done in a way that makes these states discrete and countable).

Statistical assembly and probabilities

It is quite impossible to predict whether a macroscopic system will be found in a particular one of its many basic states. However, it is possible to make statistical predictions. Instead of considering a single such macroscopic system, one then needs to consider a statistical assembly of a large number $\mathcal{N}$ of such macroscopic systems, all observed under the same known conditions. One may then try to predict the fraction $\mathcal{N}_r/\mathcal{N}$ of these systems in any particular basic state, i.e., the probability P_r that a system is in any such state. Furthermore, one can measure this probability by suitable observations of the statistical assembly.

Example

A gas of N molecules is initially confined by a partition to the left half of a box. Suppose that one is interested in studying what happens when the partition is suddenly removed so that the gas is now free to fill the entire box. To achieve a statistical description of this situation, one would consider a statistical assembly consisting of a large number $\mathcal{N}$ of such boxes — under conditions where the gas molecules in each box are initially in its left half. (See Fig. B-2.) One could then try to determine what fraction of the gases in this assembly would be in any particular state at any subsequent time, i.e., to determine the *probability* of finding the gas in any such state. (This probability would, of course, vary in the course of time as the gas spreads out throughout the entire box.)

Statistical postulate

In order to make theoretical statistical predictions about macroscopic systems, one must introduce some fundamental statistical postulates. The following examples suggest such a postulate.

Suggestive examples

Tossing coins. The two sides of an ordinary coin are mechanically indistinguishable (e.g., neither side is heavier than the other). If a set of such coins is tossed, nothing in the laws of mechanics would predict that any coin should preferentially land on one of its sides rather than on the other. Hence it is plausible to assume that such a set of tossed coins will be equally likely to land in any one of its possible states.

Positions of gas molecules. Consider the molecules of a gas contained in a box (like any of the boxes illustrated in Fig. B-2). Each molecule must then remain inside the box (i.e., the only possible positions of a molecule are those located *inside* the box). However, except for this restriction, the gas molecules can move randomly throughout the box, occasionally colliding with each other or with the walls of the box.

Nothing in the laws of mechanics would predict that a molecule should preferentially be located at any position inside the box rather than at any other. Hence one would expect that, after the molecules have moved around in the box a long time, the gas will be equally likely to be found in each of its possible basic states (corresponding to the different possible positions of its molecules). Furthermore, this should be true irrespective of the initial positions of the molecules (e.g., even if all the molecules happened initially to be located in only one part of the box).

The effects of the gravitational forces on the molecules by the earth can be assumed to be negligibly small.

Velocities of gas molecules. Consider now the velocities of a gas contained in a box. If the box is isolated, the total energy of all the gas molecules must remain unchanged. Correspondingly, the velocities of the molecules cannot have all conceivable values, but only those values consistent with the total energy. However, except for this mild restriction, the velocities of the molecules can have all possible values as the molecules move around within the box.

Once again, nothing in the laws of mechanics would predict that a molecule should preferentially have one particular velocity rather than another. Hence one would expect that, after the molecules have moved around in the box a long time, the gas will be equally likely to be found in each of its possible basic states (corresponding to the possible different velocities of its molecules). Again, this should be true irrespective of the initial velocities of the molecules.

Statistical equilibrium postulate. The preceding examples can be generalized to apply to any isolated macroscopic system. Any such system can be in any one of a very large number of *possible* basic states (i.e., states consistent with the known information about the system). But nothing in the laws of mechanics would predict that the system should be found preferentially in any one of these basic states rather than in any other. Hence we expect that, after the particles in the system have moved about for a sufficiently long time, the system should be equally likely to be found in any one of its basic states. Furthermore, this should be true irrespective of how the system started out initially.

This expectation suggests the following fundamental statistical postulate for macroscopic systems:

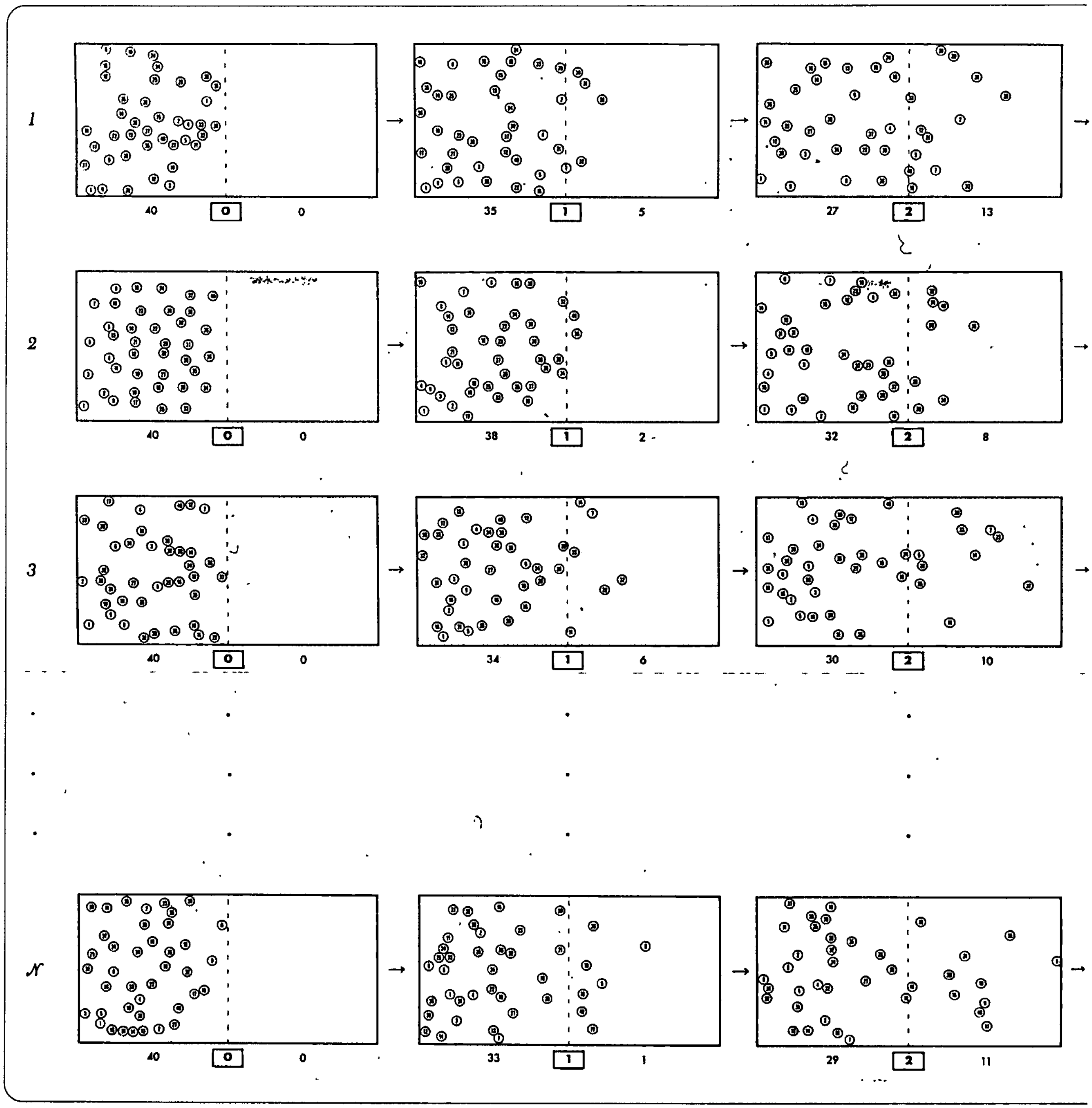
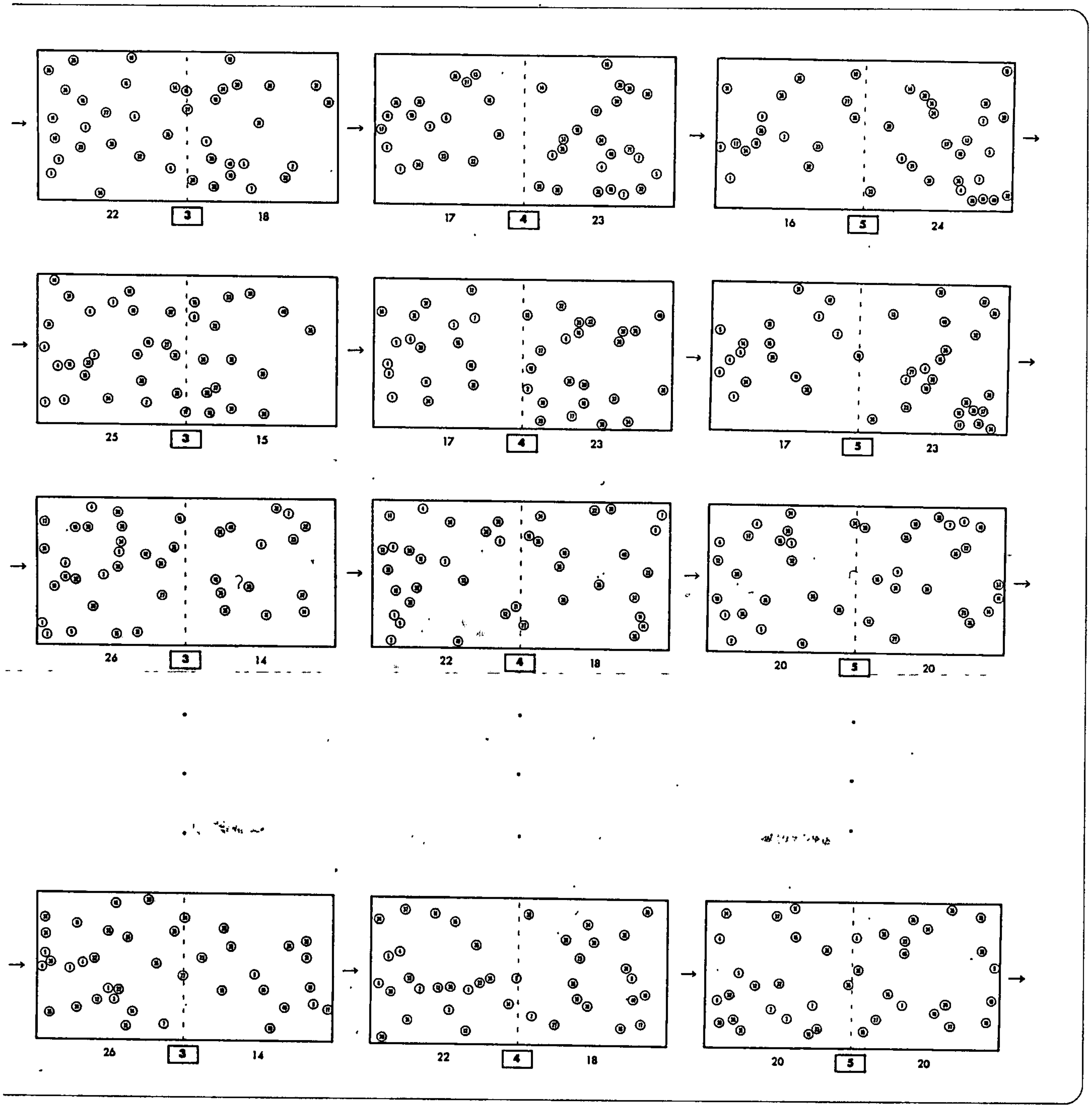


Fig. B-2. Schematic diagram showing a statistical assembly of N boxes of a gas initially confined to the left half of the box. The successive movie frames displayed in each horizontal row show what happens to each system in the assembly in the course of time. By observing all the boxes in the assembly at any one of these times (i.e., by considering all the movie frames in a column), one can determine the probability of finding the gas in any particular state at this time. [The figure was produced by a computer-generated simulation of a gas of 40 molecules colliding with each other and with the walls of the box.]


22 3 18
17 4 23
16 5 24
25 3 15
17 4 23
17 5 23
26 3 14
22 4 18
20 5 20
26 3 14
22 4 18
20 5 20

> ***Statistical equilibrium postulate:*** The probabilities of finding an isolated macroscopic system in its possible basic states change with time until they reach an unchanging equilibrium where they are equal. (B-2)

The statistical postulate (B-2) has been suggested by various examples and plausible arguments. Its ultimate validity must, of course, be assessed by how well its predicted consequences lead to conclusions well-verified by observations. Indeed, the seemingly simple statistical postulate (B-2) leads to an impressively large number of correct predictions about all kinds of macroscopic systems. This postulate provides, therefore, the basis for everything else discussed throughout the rest of the book.

Statistical calculations

The statistical postulate (B-2) allows one to make a wide variety of inferences about macroscopic systems.

Equilibrium situations. The postulate makes a definite quantitative statement about time-independent equilibrium situations, i.e., it asserts that in such a situation the system is equally like to be in any of its possible basic states. Hence the postulate allows one to make quantitative calculations about the equilibrium properties of any macroscopic system.

Non-equilibrium situations. The statistical postulate (B-2) makes only a qualitative statement about non-equilibrium situations, i.e., it asserts that such a situation will change until equilibrium is ultimately reached. Hence the postulate allows one to make qualitative predictions about whether macroscopic systems will change and about the *direction* of this change (i.e., the change is toward an equilibrium situation).

However, the postulate does *not* make any statements about how rapidly any such change occurs. Calculations of *rates* of change thus require more complex and detailed considerations than those provided by the postulate (B-2).

The following chapters will discuss these important kinds of conclusions implied by the statistical postulate (B-2).

Problems

The following problems illustrate the ideas of this section in some situations that are unrealistically simple compared to those involving macroscopic systems consisting of some 10^{24} atomic particles. However, they help prepare one to deal with the complexities of such real systems.

[B-1] ***Analogy of card shuffling***

A boy takes a deck of 52 cards and arranges them so that all the 13 hearts are at the top of the deck, the 13 diamonds are next, the 13 spades are next, and the 13 clubs are at the bottom. The boy then asks some other person to pull a card out of the deck.

(a) Suppose that the person pulls the twentieth card from the top of the deck. Is this card equally likely to be a heart, a diamond, a spade, or a club? What is the probability that it is a heart? That it is a diamond? That it is a spade? That it is a club?
(b) The boy now shuffles the deck of cards thoroughly for a long time and again give it to the other person. The person again pulls the twentieth card from the top of the deck. Is this card equally likely to be a heart, a diamond, a spade, or a club? What is the probability that this card is the king of hearts? What is the probability that it is the jack of clubs?
(c) Suppose that the boy had originally arranged the deck in some other way before shuffling the deck. Would the probabilities calculated in part *b* then be the same of different? <*a-6*>

(The shuffling of these cards is analogous to the particle motions in a macroscopic system, motions that randomize the probabilities of finding the system in any one of its possible states.)

[B-2] ***Spatial distribution of a gas of three molecules***

Consider a very simple gas consisting of only three molecules 1, 2, and 3 free to move inside the box illustrated in Fig. B-3. The position of any molecule can be specified with adequate precision by merely specifying whether it is located in the left half or the right half of the box. (For the sake of simplicity, we ignore information about the velocities of the molecules.)

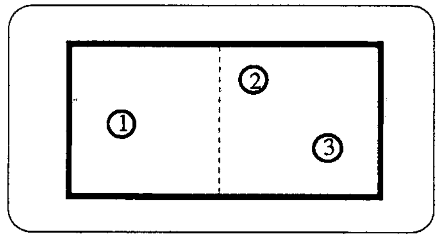


Fig. B-3. A gas consisting of three molecules in a box.

(a) Any possible basic state of the entire *gas* can then be specified by specifying whether each of its molecules is in the left half or the right half of the box. What is the total number of possible basic states of the gas?
(b) List these possible basic states by using the letter L or R to indicate whether a molecule is in the left half or the right half. (For example, LRR can indicate the state where molecule 1 is in the left half while molecules 2 and 3 are in the right half.)
(c) In how many of these basic states can the gas possibly be in a situation where all three molecules are in the left half? In how many states can it be when two molecules are in the left half (and the remaining molecule is in the right half)? In how many states can it be when one molecule is in the left half? In how many states can it be when no molecule is in the left half?
(d) If the gas in equilibrium, each of the possible basic states of the gas is equally probable. What then is the probability of a situation where all three of the molecules are found in the left half of the box? Where two of these molecules are found in the left half? Where one of these molecules is found in the left half? Where none of these molecules are found in the left half?
(e) What is the mean number of molecules in the left half of the box? <*h-4*> <*h-7*> <*a-12*>

[B-3] ***Velocity distribution of two molecules moving in one dimension***

As a very simple hypothetical example, consider a gas consisting of two molecules A and B having respective masses 2m and 4m. Suppose that these molecules are confined within a one-dimensional box within which they can move back and forth along the x-axis. The velocity component v_x of each molecule can be positive or negative, and can have the magnitudes 0, V, 2V, 3V, 4V, etc. (In other words, the values of the velocity have been made discrete by grouping together all velocities within an interval of size V.) The total energy of this gas is merely its kinetic energy since their potential energy of interaction is negligible. Since the gas is isolated, this total energy is constant and is known to be equal to $9mV^2$.

For instance, A might be a hydrogen molecule (H_2), B might be a helium molecule (He), and m might be the mass of a hydrogen atom.

(a) What is the kinetic energy of molecule A if its velocity component v_{Ax} is equal to 0? If it is equal to V? If it is equal to 2V? If it is equal to 3V?
(b) What is the kinetic energy of molecule B if its velocity component v_{Bx} is equal to 0? If it is equal to V? If it is equal to 2V? If it is equal to 3V?

(c) For the sake of simplicity, focus merely on the velocities of the molecules and ignore their positions. Each basic state of the gas can then be specified by specifying the values of the velocity components v_{AX} and v_{BX} of the two molecules.

List all the possible basic states of the gas. Specify each by the pair of values (v_{AX}, v_{BX}). (Remember that the total energy of the gas is $9mV^2$ and that each velocity component can be positive or negative.)

(d) How many such basic states are there?

(e) Suppose that the gas is in equilibrium. According to the statistical postulate (B-2), what then would be the probability of finding the gas in each one of these basic states? *<h-2>* *<a-1>*

[B-4] ***Possible positions of a single particle***

Consider a single particle which can be located anywhere inside a box. Each of the possible positions of the particle can then be indicated by one of the points indicated in Fig. B-4. [Each such point represents an elementary small range of positions (e.g., a small cube) surrounding this point. We say that the particle is at this point if it lies anywhere within this small range.]

(a) Suppose that the particle is located in a box of larger volume, as indicated in Fig. B-4b. Is the number of possible positions available to the particle then larger, smaller, or the same?

(b) Suppose that the volume of the box in Fig. B-4b is twice as large as that of the box in Fig. B-4a. How much larger than is the number of possible positions available to the particle in the larger box? *<a-9>*

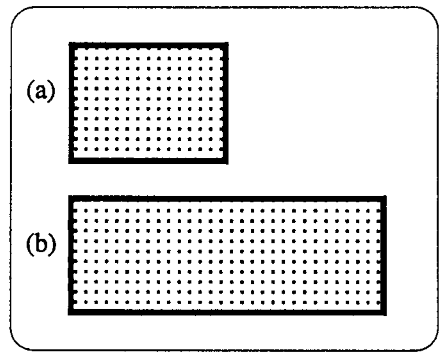


Fig. B-4. Possible positions of a particle located inside a box. (a) Original box. (b) Larger box.

C. Summary

Definitions

Statistical assembly: A very large set of similar systems described by the same known information (although differing in other respects).

Probability: The probability P_r that a system is in a state r is the fraction of systems found in this state within a very large statistical assembly of such systems.

Basic state of a macroscopic system: A basic state of a macroscopic system is the state of this system described from a detailed atomic point of view. (It may be specified by the position and velocity of each of the atomic particles in the system.)

Important knowledge

Statistical equilibrium postulate: The probabilities of finding an isolated macroscopic system in its possible basic states change with time until they reach an unchanging equilibrium situation where they are equal.

New abilities

You should now be able to do the following:

(1) Calculate probabilities in some simple situations where a system can only be in one of very few possible states.

(2) Specify the basic states of some very simple macroscopic systems and find the probabilities of these states in equilibrium.

Problems

[C-1] ***Mean score obtained in rolling two dice***

The score obtained in rolling two dice is the sum of obtained by adding the number of dots on the shown faces of the dice. In Problem A-2 you found the probabilities of obtaining any such score.

(a) Use these probabilities to calculate the mean score obtained in rolling these dice.

(b) Is this mean score larger than, smaller than, or equal to the most probable score obtained in rolling these dice? <*a-15*>

[C-2] ***Digitized quantities in computer applications***

Digitizing information (as was done for positions and velocities in Sec. B) is useful in many domains. For example, computers handle all their information in digital form (so that this information is ultimately represented by integers). All continuous quantities must, therefore, be digitized before they can be processed by a computer.

(a) For example, pictures in a plane consist of an infinite number of points. But on a computer display this position information must be digitized. Accordingly, a small range of neighboring positions is lumped together to form a single picture element ("pixel"). The whole picture can then be described by information about a finite number of such discrete pixels. A pixel is typically a small square about 0.30 mm on a side. What then is the total number of pixels contained in a rectangular computer display which is 30.0 cm wide and 22.5 cm high?

(b) The loudness of a sound can be specified by its *amplitude*, a continuous quantity. To digitize a sound played by a computer, one again needs to subdivide the amplitude into a finite number of small elementary ranges. For example, a computer playing "8-bit sound" subdivides the maximum sound amplitude into 2^8 such small ranges. How many distinguishable loudness levels are provided by such a computer? <*a-13*>

[C-3] ***Die used in an elimination game***

A person in a game is allowed to roll a die repeatedly, but is eliminated from the game when rolling an "ace" (i.e., when getting the face showing a single dot). Assume the game is played with a fair die equally likely to show any of its faces.

(a) What is the probability that the person is eliminated when rolling the die the first time?

(b) What is the probability that the person can go on to roll the die a second time?

(c) What is the probability of staying in the game after rolling the die once and of then being eliminated from the game when rolling the die the second time?

(d) What is the probability of still remaining in the game after rolling the die two times?

(e) What is the probability of being eliminated before having the opportunity to roll the die a third time? <*h-6*> <*a-11*>

[C-4]† ***Possible velocities of a single particle***

Consider a single particle free to move in three-dimensional space. The particle's three velocity components v_x, v_y, and v_z can be indicated on a graph, like that of Fig. C-1, whose axes are labeled by these three components. The identifiable velocities of the particle can then be indicated by the points indicated in Fig. B-5 . [Each such point represents an elementary small range of velocities. We say that the particle has the velocity indicated by this point if its velocity components lie in this small range.]

The speed v of a particle is related to its velocity components so that

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

Particles having this speed are thus those which are located on a sphere of radius v in the graph of Fig. C-1. Suppose that a particle is known to have a kinetic energy in some small range, and thus to have a speed in some small range between v and v + dv. Then all the possible velocities of the particle are those indicated by the dots lying in the spherical shell between the sphere of radius v and the sphere of radius v + dv.

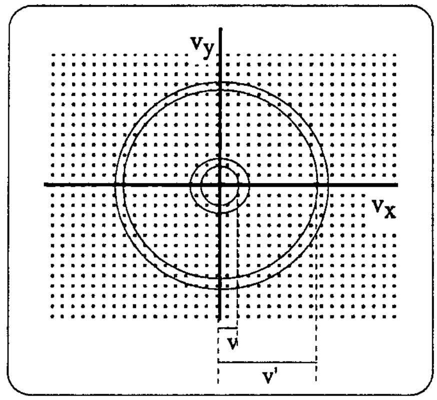


Fig. C-1 A graph whose axes indicate the three velocity components v_x, v_y, and v_z of a particle. (The axis indicating the v_z component is not shown, but points out of the paper.) The possible velocities of the particles are indicated by the dots.

(a) Suppose that the speed of the particle has a value v' larger than its original value v (as indicated in Fig. C-1) Is the number of possible particle velocities in the range of speeds between v' and v' + dv larger, smaller, or the same as the number of possible particle velocities in the original range between v and v + dv?

(b) The number of identifiable velocities with speeds less than v is proportional to the volume of the sphere of radius v. How large is this volume? <*h-12*>

(c) The number of identifiable velocities with speeds less than v + dv is proportional to the volume of the sphere of radius v + dv. How large is this volume?

(d) The number of possible velocities with speeds in the range between v and v + dv is then proportional to the volume of the spherical shell between these spheres, i.e., to the difference between the volumes of these two slightly different spheres. How large is the volume of this spherical shell? <*h-8*>

(e) Suppose that the particle's speed v is doubled (while the small speed range dv remains the same). How much larger than is the number of possible velocities which the particle can have? <*a-8*>